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Nonexistence of positive solutions of an integral system with weights

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Abstract

In this article, we study nonexistence, radial symmetry, and monotonicity of the positive solutions for a class of integral systems with weights. We use a new type of moving plane method introduced by Chen-Li-Ou. Our new ingredient is the use of Hardy-Littlewood-Sobolev inequality instead of Maximum Principle. Our results are new even for the Laplace case.

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Keywords: integral system, moving plane method, nonexistence, radial symmetry and monotonicity

1. Introduction

In this article, we study positive solutions of the following system of integral equations in $\mathbb{R}^N (N \geq 3)$,

$$\begin{cases} u(x) = \int\limits_{\mathbb{R}^N} \dfrac{v(y)^q}{|\,y|^{\xi}\,|\,x-y|^{N-\alpha}} \mathrm{d}y, \\ v(x) = \int\limits_{\mathbb{R}^N} \dfrac{u(y)^p}{|\,y|^{\eta}\,|\,x-y|^{N-\alpha}} \mathrm{d}y, \end{cases} \tag{1.1}$$

with ξ, η <0, 0 <α <N, $1 < p \leq \frac{N+\alpha-\eta}{N-\alpha}$ and $1 < q \leq \frac{N+\alpha-\xi}{N-\alpha}$. Under certain restrictions of regularity, the non-negative solution (u, v) of (1.1) is proved to be trivial or radially symmetric with respect to some point of $\mathbb{R}^N$ respectively.

The integral system (1.1) is closely related to the system of PDEs in $\mathbb{R}^N$

$$\begin{cases} (-\Delta)^{\alpha/2} u = \dfrac{v^q}{|\,x|^{\xi}}, \\ (-\Delta)^{\alpha/2} v = \dfrac{u^p}{|\,x|^{\eta}}. \end{cases} \tag{1.2}$$

In fact, every positive smooth solution of PDE (1.2) multiplied by a constant satisfies (1.1). This equivalence between integral and PDE systems for $\alpha = 2$ can be verified as in the proof of Theorem 1 in [1]. For single equations, we refer to [[2], Theorem 4.1]. Here, in (1.2), the following definition is used.

$$(-\Delta)^{\alpha/2} u = (|\,\chi|^{\alpha} u^{\wedge})^{\vee}$$

where $\wedge$ is the Fourier transformation and $\vee$ its inverse.

When $\alpha = 2$, Figueiredo et al. [3] studied the system of PDEs (1.2) in a bounded smooth domain Ω with Dirichlet boundary conditions. They found a critical hyperbola, given by

$$\frac{N-\xi}{q+1} + \frac{N-\eta}{p+1} = N-2, \quad p,\ q > 0. \tag{1.3}$$

Below this hyperbola they showed the existence of nontrivial solutions of (1.2). Interestingly, this hyperbola is closely related to the problem (1.2) in the whole space. For $\alpha = 2$ and ξ, $\eta = 0$, i.e., the elliptic systems without weights in $\mathbb{R}^N$, Serrin conjectured that (1.2) has no bounded positive solutions below the hyperbola of (1.3). It is known that above this hyperbola, (1.2) has positive solutions. Some Liouville-type results were shown in [4,5] (see also [6,7]).

When $\alpha = 2$ and ξ, $\eta \leq 0$, Felmer [8] proved the radial symmetry of the solutions of the corresponding elliptic system (1.2) by the moving plane method which was based on Maximum Principle, going back to Alexandroff, Serrin [9], and Gidas et al. [10].

For ξ, $\eta > 0$, Chen and Li [11] proved the radial symmetry of solutions of (1.1) on the hyperbola (1.3). In the special case, when ξ, $\eta = 0$, the system (1.1) reduces to

$$\begin{cases} u(x) = \int\limits_{\mathbb{R}^N} \dfrac{v(y)^q}{|x-y|^{N-\alpha}} \mathrm{d}y, \\ v(x) = \int\limits_{\mathbb{R}^N} \dfrac{u(y)^p}{|x-y|^{N-\alpha}} \mathrm{d}y. \end{cases} \tag{1.4}$$

The integral system (1.4) is closely related to the system of PDEs

$$\begin{cases} (-\Delta)^{\alpha/2} u = v^q, \\ (-\Delta)^{\alpha/2} v = u^p. \end{cases} \tag{1.5}$$

Recently, using the method of moving planes, Ma and Chen [12] proved a Liouville-type theorem of (1.4), and for the more generalized system,

$$\begin{cases} u(x) = \int\limits_{\mathbb{R}^N} \dfrac{v(y)^q}{|x-y|^{N-\alpha}} \mathrm{d}y, \\ v(x) = \int\limits_{\mathbb{R}^N} \dfrac{u(y)^p}{|x-y|^{N-\beta}} \mathrm{d}y. \end{cases} \tag{1.6}$$

Huang et al. [13] proved the existence, radial symmetry and monotonicity under some assumptions of p, q, α, and β. Furthermore, using Doubling Lemma indicated in [14], which is an extension of an idea of Hu [15], Chen and Li [[16], Theorem 4.3] obtained the nonexistence of positive solutions of (1.4) under some stronger integrability conditions (e.g., $u, v \in L^\infty_{loc}$ are necessary). In fact, for System (1.5) of $\alpha = 2$, Liouville-type theorems are known for (q, p) in the region $[0, \frac{N+2}{N-2}] \times [0, \frac{N+2}{N-2}]$. For the interested readers, we refer to [17,18] and their generalized cases [19,20], where the results were proved by the moving plane method or the method of moving spheres which both deeply depend on Maximum Principle. In [21], Mitidieri proved that if (q, p) satisfies

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \quad p,\ q > 0, \tag{1.7}$$

then System (1.5) possesses no nontrivial radial positive solutions. Later, Mitidieri [22] showed that a Liouvillle-type theorem holds if (q, p) satisfies

$$\frac{N-2}{N} \le \max\left\{\frac{q+1}{qp-1}, \frac{p+1}{qp-1}\right\},$$

generalizing a work by Souto [23]. In [24], Serrin and Zou proved that for (q, p) satisfying (1.7), there exists no positive solution of System (1.1) when the solution has an appropriate decay at infinity.

When $\alpha = 2$, it has been conjectured that a Liouville-type theorem of System (1.5) holds if the condition (1.7) holds. This conjecture is further suggested by the works of Van der Vorst [25] and Mitidieri [21] on existence in bounded domains, Hulshof and Van der Vorst [26], Figueiredo and Felmer [6] on existence on bounded domains through variational method, and Serrin and Zou [27] on existence of positive radial solutions when the inequality in (1.7) is reversed. Figueierdo and Felmer [17], Souto [28], and Serrin and Zou [24] studied System (1.5) and obtained some Liouville-type results. Ma and Chen [12] gave a partial generalized result about their work. Serrin conjectured that if (q, p) satisfies (1.7), System (1.5) has no bounded positive solutions. It is known that outside the region of (1.7), System (1.5) has positive solutions. We believe that the critical hyperbola in the conjecture is closely related to the famous Hardy-Littlewood-Sobolev inequality [29] and its generalization. For more results about elliptic systems, one may look at the survey paper of Figueierdo [30].

There are some related works about this article. When $u(x) = v(x)$ and $q = p = \frac{N+\alpha}{N-\alpha}$, System (1.4) becomes the single equation

$$u(x) = \int_{\mathbb{R}^N} \frac{u(y)^{\frac{N+\alpha}{N-\alpha}}}{|x-y|^{N-\alpha}} dy, \quad u > 0 \text{ in } \mathbb{R}^N. \tag{1.8}$$

The corresponding PDE is the well-known family of semilinear equations

$$(-\Delta)^{\alpha/2} u = u^{\frac{N+\alpha}{N-\alpha}}, \quad u > 0 \text{ in } \mathbb{R}^N. \tag{1.9}$$

In particular, when $N \ge 3$ and $\alpha = 2$, (1.9) becomes

$$-\Delta u = u^{\frac{N+2}{N-2}}, \quad u > 0 \text{ in } \mathbb{R}^N. \tag{1.10}$$

The classification of the solutions of (1.10) has provided an important ingredient in the study of the well-known Yamabe problem and the prescribing scalar curvature problem. Equation (1.10) was studied by Gidas et al. [31], Caffarelli et al. [32], Chen and Li [33] and Li [34]. They classified all the positive solutions. In the critical case, Equation (1.10) has a two-parameter family of solutions given by

$$u(x) = \left(\frac{c}{d + |x-\bar{x}|^2}\right)^{\frac{N-2}{2}}, \tag{1.11}$$

where $c = [N(N-2)d]^{\frac{1}{2}}$ with $d > 0$ and $\bar{x} \in \mathbb{R}^N$. Recently, Wei and Xu [35] generalized this result to the solutions of the more general Equation (1.9) with α being any even number between 0 and N.

Apparently, for other real values of α between 0 and N, (1.9) is also of practical interest and importance. For instance, it arises as the Euler-Lagrange equation of the functional

$$I(u) = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{4}} u|^2 \mathrm{d}x \Big/ \left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-\alpha}} \mathrm{d}x \right)^{\frac{N-\alpha}{N}}.$$

The classification of the solutions would provide the best constant in the inequality of the critical Sobolev imbedding from $H^{\frac{\alpha}{2}}(\mathbb{R}^N)$ to $L^{\frac{2N}{N-\alpha}}(\mathbb{R}^N)$:

$$\left(\int_{\mathbb{R}^N} |u|^{\frac{2N}{N-\alpha}} \mathrm{d}x \right)^{\frac{N-\alpha}{N}} \leq C \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\alpha}{4}} u|^2 \mathrm{d}x.$$

Let us emphasize that considerable attention has been drawn to Liouville-type results and existence of positive solutions for general nonlinear elliptic equations and systems, and that numerous related works are devoted to some of its variants, such as more general quasilinear operators and domains, and the blowup questions for nonlinear parabolic equations and systems. We refer the interested reader to [20,22,26,27,36-39], and some of the references therein.

Our results in the present article can be considered as a generalization of those in [8,12,17,18]. We note that we here use the Kelvin-type transform and a new type of moving plane method introduced by Chen-Li-Ou, and our new ingredient is the use of Hardy-Littlewood-Sobolev inequality instead of Maximum Principle. Our results are new even for the Laplace case of $\alpha = 2$.

Our main results are the following two theorems.

Theorem 1.1. *Let the pair* (u, v) *be a non-negative solution of* (1.1) *and* $\frac{N-\xi}{N-\alpha} < q \leq \frac{N+\alpha-\xi}{N-\alpha}$, $\frac{N-\xi}{N-\alpha} < q \leq \frac{N+\alpha-\xi}{N-\alpha}$ *with* ξ, $\eta < 0$ *and* $0 < \alpha < N$, *but* $p = \frac{N+\alpha-\eta}{N-\alpha}$ *and* $q = \frac{N+\alpha-\xi}{N-\alpha}$ *are not true at the same time. Moreover, assume that* $u \in L^{\beta}_{loc}(\mathbb{R}^N)$ *and* $\beta = \frac{p-1}{\frac{(N-\alpha)p+\eta}{N}-1}$ *with* $\beta = \frac{p-1}{\frac{(N-\alpha)p+\eta}{N}-1}$ *and* $\phi = \frac{q-1}{\frac{(N-\alpha)q+\xi}{N}-1}$. *Then both u and v are trivial, i.e.,* $(u, v) = (0, 0)$.

Theorem 1.2. *Let the pair* (u, v) *be a non-negative solution of* (1.1) *and* $p = \frac{N+\alpha-\eta}{N-\alpha}$, $q = \frac{N+\alpha-\xi}{N-\alpha}$ *with* ξ, $\eta < 0$ *and* $0 < \alpha < N$. *Moreover, assume that* $u \in L^{\beta}_{loc}(\mathbb{R}^N)$ *and* $\beta = \frac{(2\alpha-\eta)N}{\alpha(N-\alpha)}$ *with* $\beta = \frac{(2\alpha-\eta)N}{\alpha(N-\alpha)}$ *and* $\phi = \frac{(2\alpha-\xi)N}{\alpha(N-\alpha)}$. *Then, u and v are radially symmetric and decreasing with respect to some point of* $\mathbb{R}^N$.

Remark 1.1. *Due to the technical difficulty, we here only consider the nonexistence and symmetry of positive solutions in the range of* ξ, $\eta < 0$, $p > \frac{N-\eta}{N-\alpha}$ *and* $q > \frac{N-\xi}{N-\alpha}$. *For* ξ, $\eta > 0$, *Chen and Li* [11]*proved the radial symmetry of solutions of* (1.1) *on the hyperbola* (1.3). *For* $\xi = \eta = 0$ *and* $\max\{1, 2/(N-2)\} < p, q < \infty$, *Chen and Li [*[16], *Theorem 4.3] obtained the nonexistence of positive solutions of* (1.1) *under some stronger integrability conditions* *(e.g., $u, v \in L^{\infty}_{loc}$ are necessary). We note that there exist many open questions on nonexistence and symmetry of positive solutions of the equation with weights as* (1.1) *in the rest range of p, q, ξ, and η. It is an interesting research subject in the future.*

We shall prove Theorem 1.1 via the Kelvin-type transform and the moving plane method (see [2,40,41]) and prove Theorem 1.2 by the similar idea as in [17].

Throughout the article, C will denote different positive constants which depend only on N, p, q, α and the solutions u and v in varying places.

2. Kelvin-type transform and nonexistence

In this section, we use the moving plane method to prove Theorem 1.1. First, we introduce the Kelvin-type transform of u and v as follows, for any $x \neq 0$,

$$\bar{u}(x) = |x|^{\alpha-N} u\left(\frac{x}{|x|^2}\right) \quad \text{and} \quad \bar{v}(x) = |x|^{\alpha-N} v\left(\frac{x}{|x|^2}\right).$$

Then by elementary calculations, one can see that (1.1) and (1.2) are transformed into the following forms:

$$\begin{cases} \bar{u}(x) = \int\limits_{\mathbb{R}^N} \dfrac{\bar{v}(y)^q}{|y|^s\,|x-y|^{N-\alpha}} \mathrm{d}y, \\ \bar{v}(x) = \int\limits_{\mathbb{R}^N} \dfrac{\bar{u}(y)^p}{|y|^t\,|x-y|^{N-\alpha}} \mathrm{d}y, \end{cases} \tag{2.1}$$

and

$$\begin{cases} (-\Delta)^{\alpha/2}\bar{u} = |x|^{-s}\bar{v}^q, \\ (-\Delta)^{\alpha/2}\bar{v} = |x|^{-t}\bar{u}^p, \end{cases} \tag{2.2}$$

where $t = (N + \alpha) - \eta - (N - \alpha)p \geq 0$ and $s = (N + \alpha) - \xi - (N - \alpha)q \geq 0$. Obviously, both $\bar{u}(x)$ and $\bar{v}(x)$ may have singularities at origin. Since $u \in L^{\beta}_{loc}(\mathbb{R}^N)$ and $v \in L^{\phi}_{loc}(\mathbb{R}^N)$, it is easy to see that $\bar{u}(x)$ and $\bar{v}(x)$ have no singularities at infinity, i.e., for any domain Ω that is a positive distance away from the origin,

$$\int\limits_{\Omega} \bar{u}(y)^{\beta} \mathrm{d}y < \infty \quad \text{and} \quad \int\limits_{\Omega} \bar{v}(y)^{\phi} \mathrm{d}y < \infty. \tag{2.3}$$

In fact, for $y = z/|z|^2$, we have

$$\begin{aligned} \int_{\Omega} \bar{u}(y)^{\beta} \mathrm{d}y &= \int\limits_{\Omega} \left(|y|^{\alpha-N} u\left(\frac{y}{|y|^2}\right)\right)^{\beta} \mathrm{d}y \\ &= \int\limits_{\Omega^*} \left(|z|^{N-\alpha} u(z)\right)^{\beta} |z|^{-2N} \mathrm{d}z \\ &= \int\limits_{\Omega^*} |z|^{\beta(N-\alpha)-2N} u(z)^{\beta} \mathrm{d}z \\ &\leq C \int\limits_{\Omega^*} u(z)^{\beta} \mathrm{d}z \\ &< \infty. \end{aligned}$$

For the second equality, we have made the transform $y = z/|z|^2$. Since Ω is a positive distance away from the origin, Ω^*, the image of Ω under this transform, is bounded. Also, note that $\beta(N - \alpha) - 2N > 0$ by the assumptions of Theorem 1.1. Then, we get the estimate (2.3).

For a given real number λ, define

$$\Sigma_\lambda = \{x = (x_1, \ldots, x_n) \mid x_1 \geq \lambda\}.$$

Let $x^\lambda = (2\lambda - x_1, x_2,\ldots, x_n)$, $\bar{u}_\lambda(x) = \bar{u}(x^\lambda)$ and $\bar{v}_\lambda(x) = \bar{v}(x^\lambda)$.

The following lemma is elementary and is similar to Lemma 2.1 in [2].

Lemma 2.1. *For any solution* $(\bar{u}(x), \bar{v}(x))$ *of (2.1), we have*

$$\bar{u}_\lambda(x) - \bar{u}(x) = \int_{\Sigma_\lambda} (|x-y|^{\alpha-N} - |x^\lambda - y|^{\alpha-N})[|y^\lambda|^{-s}\bar{v}_\lambda(y)^q - |y|^{-s}\bar{v}(y)^q]\,\mathrm{d}y \tag{2.4}$$

and

$$\bar{v}_\lambda(x) - \bar{v}(x) = \int_{\Sigma_\lambda} (|x-y|^{\alpha-N} - |x^\lambda - y|^{\alpha-N})[|y^\lambda|^{-t}\bar{u}_\lambda(y)^p - |y|^{-t}\bar{u}(y)^p]\,\mathrm{d}y. \tag{2.5}$$

Proof. It is easy to see that

$$\begin{aligned}\bar{u}(x) &= \int_{\Sigma_\lambda} |y|^{-s}\,|x-y|^{\alpha-N}\bar{v}^q(y)\mathrm{d}y \\ &+ \int_{\Sigma_\lambda} |y^\lambda|^{-s}\,|x^\lambda - y|^{\alpha-N}\overline{v_\lambda}^q(y)\mathrm{d}y.\end{aligned} \tag{2.6}$$

Substituting x by x^λ, we have

$$\begin{aligned}\bar{u}(x^\lambda) &= \int_{\Sigma_\lambda} |y|^{-s}\,|x^\lambda - y|^{\alpha-N}\bar{v}^q(y)\mathrm{d}y \\ &+ \int_{\Sigma_\lambda} |y^\lambda|^{-s}\,|x-y|^{\alpha-N}\overline{v_\lambda}^q(y)\mathrm{d}y.\end{aligned} \tag{2.7}$$

The fact that $|x - y^\lambda| = |x^\lambda - y|$ implies (2.4). Similarly, one can show that (2.5) holds. So, Lemma 2.1 is proved.

Proof of Theorem 1.1.

Outline: Let x_1 and x_2 be any two points in $\mathbb{R}^N$. We shall show that

$$u(x_1) = u(x_2) \quad \text{and} \quad v(x_1) = v(x_2)$$

and therefore u and v must be constants. This is impossible unless $u = v = 0$. To obtain this, we show that u and v are symmetric about the midpoint $(x_1 + x_2)/2$. We may assume that the midpoint is at the origin. Let $\bar{u}$ and $\bar{v}$ be the Kelvin-type transformations of u and v, respectively. Then, what left to prove is that $\bar{u}$ and $\bar{v}$ are symmetric about the origin. We shall carry this out in the following three steps.

Step 1. Define

$$\Sigma_\lambda^{\bar{u}} = \{x \in \Sigma_\lambda \mid \bar{u}(x) < \bar{u}_\lambda(x)\}$$

and

$$\Sigma_\lambda^{\bar{v}} = \{x \in \Sigma_\lambda \mid \bar{v}(x) < \bar{v}_\lambda(x)\}$$

We show that for sufficiently negative values of λ, both $\Sigma_\lambda^{\bar{u}}$ and $\Sigma_\lambda^{\bar{v}}$ must be empty.

Whenever $x, y \in \Sigma_\lambda$, we have that $|x - y| \le |x^\lambda - y|$. Moreover, since $\lambda < 0$, $|y^\lambda| \ge |y|$ for any $y \in \Sigma_\lambda$. Then by emma 2.1, for any $x \in \Sigma_\lambda$, it is easy to verify that

$$\begin{aligned} \bar{u}_\lambda(x) - \bar{u}(x) &\le \int_{\Sigma_\lambda} (|x-y|^{\alpha-N} - |x^\lambda - y|^{\alpha-N})\,|y|^{-s}[\bar{v}_\lambda(y)^q - \bar{v}(y)^q]\mathrm{d}y \\ &\le \int_{\Sigma_\lambda^{\bar{v}}} |x-y|^{\alpha-N}\,|y|^{-s}[\bar{v}_\lambda(y)^q - \bar{v}(y)^q]\mathrm{d}y \\ &\le \int_{\Sigma_\lambda^{\bar{v}}} |x-y|^{\alpha-N}\,|y|^{-s}[\bar{v}_\lambda(y)^{q-1}(\bar{v}_\lambda(y) - \bar{v}(y))]\mathrm{d}y. \end{aligned} \tag{2.8}$$

Now we recall the double weighted Hardy-Littlewood-Sobolev inequality which was generalized by Stein and Weiss [42]:

$$\left\| \int \frac{f(y)}{|x|^\gamma\,|x-y|^\lambda\,|y|^\tau}\mathrm{d}y \right\|_{\bar{q}} \le C_{\gamma,\tau,\bar{p},\lambda,N}\,\|f\|_{\bar{p}}, \tag{2.9}$$

where $0 \le \tau < N/\bar{p}'$, $0 \le \gamma < N/\bar{q}$ and $1/\bar{p} + (\gamma + \tau + \lambda)/N = 1 + 1/\bar{q}$ with $1/\bar{p} + 1/\bar{p}' = 1$.

It follows first from inequality (2.9) and then the Hölder inequality that, for any $r > \max\{(N - \xi)/(N - \alpha), (N - \eta)/(N - \alpha)\}$,

$$\begin{aligned} \|\bar{u}_\lambda - \bar{u}\|_{L^r(\Sigma_\lambda^{\bar{u}})} &\le C \left\| \int_{\Sigma_\lambda^{\bar{v}}} |x-y|^{\alpha-N}\,|y|^{-s}[\bar{v}_\lambda(y)^{q-1}(\bar{v}_\lambda(y) - \bar{v}(y))]\mathrm{d}y \right\|_{L^r(\Sigma_\lambda^{\bar{v}})} \\ &\le C\,\|\bar{v}_\lambda\|^{q-1}_{L^\phi(\Sigma_\lambda^{\bar{v}})}\,\|\bar{v}_\lambda(y) - \bar{v}(y)\|_{L^r(\Sigma_\lambda^{\bar{v}})}, \end{aligned} \tag{2.10}$$

where $\phi = \frac{q-1}{\frac{(N-\alpha)q+\xi}{N} - 1}$.

Similarly, one can show that

$$\|\bar{v}_\lambda - \bar{v}\|_{L^r(\Sigma_\lambda^{\bar{v}})} \le C\,\|\bar{u}_\lambda\|^{p-1}_{L^\beta(\Sigma_\lambda^{\bar{u}})}\,\|\bar{u}_\lambda(y) - \bar{u}(y)\|_{L^r(\Sigma_\lambda^{\bar{u}})}, \tag{2.11}$$

where $\beta = \frac{p-1}{\frac{(N-\alpha)p+\eta}{N} - 1}$.

Combining (2.10) and (2.11), we arrive at

$$\|\bar{u}_\lambda - \bar{u}\|_{L^r(\Sigma_\lambda^{\bar{u}})} \le C\,\|\bar{v}_\lambda\|^{q-1}_{L^\phi(\Sigma_\lambda^{\bar{v}})}\,\|\bar{u}_\lambda\|^{p-1}_{L^\beta(\Sigma_\lambda^{\bar{u}})}\,\|\bar{u}_\lambda - \bar{u}\|_{L^r(\Sigma_\lambda^{\bar{u}})}. \tag{2.12}$$

By the integrability conditions, we can choose M sufficiently large, such that for $\lambda \le -M$, we have

$$C\,\|\bar{v}_\lambda\|^{q-1}_{L^\phi(\Sigma_\lambda^{\bar{v}})}\,\|\bar{u}_\lambda\|^{p-1}_{L^\beta(\Sigma_\lambda^{\bar{u}})} \le \frac{1}{2}. \tag{2.13}$$

These imply that $\| \bar{u}_\lambda - \bar{u}\|_{L^r(\Sigma_\lambda^{\bar{u}})} = 0$. In other words, $\Sigma_\lambda^{\bar{u}}$ must be measure zero, and hence empty. Similarly, one can show that $\Sigma_\lambda^{\bar{v}}$ is empty. Step 1 is complete.

Step 2. Now we have that for $\lambda \leq -M$,

$$\bar{u}(x) \geq \bar{u}_\lambda(x) \quad \text{and} \quad \bar{v}(x) \geq \bar{v}_\lambda(x), \quad \forall x \in \sum\nolimits_\lambda. \tag{2.14}$$

Thus, we can move the plane $\lambda \leq -M$ to the right as long as (2.14) holds. Suppose that at one $\lambda_0 < 0$, we have, on $\sum_\lambda$

$$\bar{u}(x) \geq \bar{u}_{\lambda_0}(x) \quad \text{and} \quad \bar{v}(x) \geq \bar{v}_{\lambda_0}(x).$$

But either

$$meas\,\{x \in \sum\nolimits_{\lambda_0} \mid \bar{u}(x) > \bar{u}_{\lambda_0}(x)\} > 0$$

or

$$meas\,\{x \in \sum\nolimits_{\lambda_0} \mid \bar{v}(x) > \bar{v}_{\lambda_0}(x)\} > 0.$$

Then, we want to show that the plane can be moved further to the right, i.e., there exists an ε depending on N, p, q and the solution $(\bar{u}\,,\ \bar{v}\,)$ such that (2.14) holds on $\sum_\lambda$ for all $\lambda \in\ [\lambda_0, \lambda_0 + \varepsilon)$.

Assume that $meas\{x \in \sum_{\lambda_0} \mid \bar{v}(x) > \bar{v}_{\lambda_0}(x)\} > 0$. By (2.4), we know that $\bar{u}(x) > \bar{u}_{\lambda_0}(x)$ in the interior of $\sum_{\lambda_0}$. Define $\widehat{\sum_{\lambda_0}^{\bar{u}}} = \{x \in \sum_{\lambda_0} \mid \bar{u}(x) \leq \bar{u}_{\lambda_0}(x)\}$ and $\widehat{\sum_{\lambda_0}^{\bar{v}}} = \{x \in \sum_{\lambda_0} \mid \bar{v}(x) \leq \bar{v}_{\lambda_0}(x)\}$. It is clear that $\widehat{\sum^{\bar{v}_\lambda}}$ has measure zero, and $\lim_{\lambda \to \lambda_0} \sum_\lambda^{\bar{u}} \subset \widehat{\sum_{\lambda_0}^{\bar{u}}}$ in the sense of measures. The same conclusion holds for $\bar{v}$. Let G^* be the reflection of the set G about the plane $x_1 = \lambda$. We see from (2.10) and (2.11) that

$$\| \bar{u}_\lambda - \bar{u}\|_{L^r(\Sigma_\lambda^{\bar{u}})} \leq C \| \bar{v}_\lambda \|^{q-1}_{L^\phi((\Sigma_\lambda^{\bar{v}})^*)} \| \bar{u}_\lambda \|^{p-1}_{L^\beta((\Sigma_\lambda^{\bar{u}})^*)} \| \bar{u}_\lambda - \bar{u}\|_{L^r(\Sigma_\lambda^{\bar{u}})}. \tag{2.15}$$

Again, the integrability of $\bar{u}$ and $\bar{v}$ ensures that one can choose ε small enough, such that for all $\lambda \in\ [\lambda_0, \lambda_0 + \varepsilon)$,

$$C \| \bar{v}_\lambda \|^{q-1}_{L^\phi((\Sigma_\lambda^{\bar{v}})^*)} \| \bar{u}_\lambda \|^{p-1}_{L^\beta((\Sigma_\lambda^{\bar{u}})^*)} \leq \frac{1}{2}.$$

Now by (2.15) we have

$$\| \bar{u}_\lambda - \bar{u}\|_{L^r(\Sigma_\lambda^{\bar{u}})} = 0$$

and therefore $\Sigma_\lambda^{\bar{u}}$ is empty. A similar argument shows that $\Sigma_\lambda^{\bar{v}}$ is empty too.

Step 3. If the plane stops at $x_1 = \lambda_0$ for some $\lambda_0 < 0$, then $\bar{u}$ and $\bar{v}$ must be symmetric and monotone about the plane $x_1 = \lambda_0$. This implies that $\bar{u}$ and $\bar{v}$ have no singularity at the origin. But the equations in (2.2) tell us that this is impossible if $\bar{u}(x)$ and $\bar{v}(x)$ are nontrivial. Hence, we can move the plane to $x_1 = 0$. Then, $\bar{u}(x)$ and $\bar{v}(x)$ are symmetric about the plane origin. Then $u = v = 0$. The proof of Theorem 1.1 is complete.

3. Symmetry and monotonicity

In this section, we prove Theorem 1.2 which shows that the non-negative solutions of System (1.1) are radially symmetric and decreasing with respect to some point in $\mathbb{R}^N$.

Proof of Theorem 1.2. We show that $\bar{u}$ and $\bar{v}$ are symmetric with respect to some plane parallel $x_1 = 0$. Indeed, if $\lambda_0 < 0$, such as the steps of Theorem 1.1, we know $\bar{u}$ and $\bar{v}$ are symmetric with respect to the hyperplane $x_1 = \lambda_0$. If $\lambda_0 = 0$, we conclude that $\bar{u}_0(x) \geq \bar{u}(x)$ and $\bar{v}_0(x) \geq \bar{v}(x)$ for all $x \in \Sigma_0$. On the other hand, we perform the moving plane procedure from the right and find a corresponding $\lambda_0^r \geq 0$. If $\lambda_0^r > 0$, an analogue to Theorem 1.1 shows that $\bar{u}$ and $\bar{v}$ are symmetric with respect to the hyperplane $x_1 = \lambda_0^r$. If $\lambda_0^r = 0$, we conclude that $\bar{u}_0(x) \geq \bar{u}(x)$ and $\bar{v}_0(x) \geq \bar{v}(x)$ for all $x \in \Sigma_0$. From above we can conclude $\bar{u}$ and $\bar{v}$ are symmetric with respect to the plane $x_1 = 0$. We perform this moving plane procedure taking planes perpendicular to any direction, and for each direction $\gamma \in \mathbb{R}^N$, $|\gamma| = 1$, we can find a plane T_γ with the property that both $\bar{u}$ and $\bar{v}$ are symmetric with respect to T_γ. A simple argument shows that all these planes intersect at a single point, or $\bar{u} = \bar{v} = 0$. The proof of Theorem 1.2 is complete.

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Competing interests

The author declares that he has no competing interests.

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